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# Calculations of the Stability of Some Axisymmetric Flows Proposed as a Model of Vortex Breakdown.

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## ABSTRACT

The term "vortex breakdown" refers to the abrupt and drastic changes of structure that can sometimes occur in swirling flows. It has been conjectured that the "bubble" type of breakdown can be viewed as an axisymmetric wave traveling upstream in a primarily columnar vortex flow. In this scenario the wave's upstream progress is impeded only when it reaches a critical amplitude and it loses stability to some non-axisymmetric disturbance. We will investigate the stability of some axisymmetric wavy flows, which model vortex breakdown, to three dimensional disturbances viewing the amplitude of the wave as a bifurcation parameter. We will also look at the stability of a set of related, columnar vortex flows which are constructed by taking the two dimensional flow at a single axial location and extending it throughout the domain without variation. The method of our investigation will be to expand the perturbation velocity in a series of divergence free vectors which ensures that the continuity equation for the incompressible fluid is satisfied exactly by the computed velocity field. Projections of the stability equation onto the space of inviscid vector fields eliminates the pressure term from the equation and reduces the differential eigen problem to a generalized matrix eigen problem. Results are presented both for the one dimensional, columnar vortex flows and also for the wavy "bubble" flows.

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## **1. Introduction: Vortex Breakdown.**

The term "vortex breakdown" refers to the abrupt and drastic changes of structure that can sometimes occur in vortex flows. Observations by Peckham & Atkinson [1957] of breakdowns occurring in the leading edge vortex formed above a swept back lifting surface and a number of studies demonstrating the serious aerodynamic consequences of such events (the slopes of the lift, drag and moment curves are all altered by breakdown) stimulated early interest in the subject. Since that time the literature on vortex breakdown has burgeoned. The interested reader is referred to review articles by Hall [1972] and Leibovich [1978, 1984] for summaries both of the experimental observations that have been made and also of the theories that have been proposed to explain them.

Experimental observations are most easily made on vortex flows confined to tubes and the bulk of the available data is for such cases. In one apparatus, used by a number of researchers, water is passed radially inward through a set of guidevanes imparting swirl to the fluid which then enters axially into a test section (a frustrum of a cone of very small cone angle), by means of an annular channel formed between a bellmouth opening on the section and a centerbody whose tip is aligned with the cone axis. The boundary layer shed from the tip of the centerbody forms a well defined vortex core along the axis of the test section and dye injected through the tip allows for flow visualization.

With this type of apparatus two parameters are within the easy control of the experimenter, namely the amount of swirl imparted to the inlet flow and the volume flow rate through the tube (effectively the Reynolds number of the flow). As the Reynolds number is increased for a sufficiently large, fixed value of swirl the breakdown assumes one of two characteristic forms.

Both of these are characterized by a rapid deceleration of the axial velocity component, occurring in the axial distance on the order of one vortex core diameter, followed by the formation of a stagnation point and (in some frame of reference) a region of reversed flow along the axis. The two forms are easily distinguished in flow visualization studies as in one form, the spiral or S type breakdown, the tracer dye assumes a spiral shape rotating in the same sense as the inlet fluid, while in the other form, the bubble or B type breakdown, the dye assumes a form that looks much like a body of revolution placed in the fluid. Our interest will be in this latter form of breakdown which is sketched in Figure 1. Here we show "ideal" or averaged stream surfaces on which the fluid particles travel in helical paths. (Leibovich [1978]).

Faler & Leibovich [1977], Garg & Leibovich [1979] and the author [unpublished studies] have used the non-invasive techniques of laser doppler anemometry to measure the velocity fields both upstream and downstream of breakdown events. Outside a thin boundary layer along the tube wall the experimental data is well fitted by the analytic profiles,

$$V(r) = \frac{1}{r} Q \left( 1 - e^{-\alpha r^2} \right) \quad (1.1)$$

$$W(r) = W_1 + W_2 e^{-\alpha r^2} \quad (1.2)$$

$W$  and  $V$  being respectively the axial and azimuthal velocity components while  $W_1$ ,  $W_2$ ,  $Q$  and  $\alpha$  are all constants (representative values are given by Garg & Leibovich [1979]). The profiles apply to the downstream flow only in the mean, as the flow there fluctuates with time.

Upstream of the recirculation zone the flow is axisymmetric and steady. After breakdown the vortex core expands to two or three times its upstream size and the constant  $W_2$  in the mean axial velocity profile which had been positive upstream (jetlike flow) becomes negative (wakelike flow). Downstream, within a few vortex core diameters of the breakdown, a turbulent wake is

invariably established . This transition to turbulence "switched on" by the coherent breakdown structure provides a further incentive for its study.

Possibly motivated by the fact that the flows upstream of breakdown can be made to have a high degree of axial symmetry, most of the research to date assumes that axially symmetric processes are the important ones in vortex breakdown. As only axisymmetric disturbances can cause a change in the axial velocity component as measured on the axis and as a deceleration of this component is so pronounced in breakdown, it is clear that such disturbances play an important role. Nevertheless, all transitions occurring in vortex flows as documented by Faler [1976] are nonaxisymmetric and the flow within the bubble itself is unsteady with regular low frequency oscillations. Furthermore, the stagnation point that defines the start of the recirculation zone is not entirely fixed but wanders over a short range of the axis in a seemingly random fashion.

Leibovich [1984] proposed the following plausible scenario for the bubble type breakdown. A finite axisymmetric disturbance, triggered off downstream, moves upstream in a columnar flow that is nearly critical in the sense of Benjamin [1962]. (A supercritical flow, in this classification, allows for the upstream propagation of infinitesimal axisymmetric waves while a subcritical flow does not). Flows of the form (1.1,2) can indeed support axisymmetric dispersive waves and these can propagate upstream in some situations (Leibovich [1970], Randall & Leibovich [1973]). Moving in this direction, the cross sectional area of the tube decreases causing the wave to amplify and speed up. The conjecture is that, upon reaching some critical amplitude, these waves lose stability to a non-axisymmetric disturbance. The growth of the asymmetric mode at the expense of the axisymmetric wave, drains energy from it and this causes the wave to equilibrate at some axial location in the diverging tube, much as is seen in experiments.

Our aim is to study the stability of some inviscid, wavy axisymmetric flows to three dimensional disturbances with the amplitude of the waves viewed as a bifurcation parameter. We start with a columnar flow in cylindrical coordinates of the form  $(0, V_0(r), W_0(r))$ , (e.g. (1.1,2)). (For arbitrary  $C^1$  functions,  $V_0$  and  $W_0$ , all such flows satisfy Euler's equations). We then seek axisymmetric wavy perturbations to this flow which satisfy the equations of motion, at least approximately, for small amplitude. In terms of a stream function,  $\psi$  and a circulation,  $\kappa$  (Lamb [1932]) Leibovich [1972] found solutions to Eulers equations of the form,

$$\psi(r, x) = \psi_0(r) + \epsilon \phi(r) A(x) + O(\epsilon^2), \quad (1.3)$$

$$\kappa(r, x) = \kappa_0(r) + \epsilon \gamma(r) A(x), + O(\epsilon^2), \quad (1.4)$$

where  $x$  is a moving coordinate,

$$x = z - dt \quad (1.5)$$

and  $d$  is a constant that must found in the calculation. The velocity components are given by

$$u = -\frac{1}{r} \frac{\partial}{\partial z} \psi, \quad (1.6)$$

$$v = \frac{1}{r} \kappa, \quad (1.7)$$

$$w = \frac{1}{r} \frac{\partial}{\partial r} \psi. \quad (1.8)$$

The columnar base flow is represented by  $\psi_0(r)$  and  $\kappa_0(r)$ .

The amplitude function,  $A(z, t)$  is governed by a Korteweg de Vries equation which has both infinite and finite period solutions. The multiple scales analysis that was used to obtain these solutions is strictly valid only for long period waves which are also the most interesting solutions from a physical point of view. When doing the stability analysis we will confine our attention, for numerical reasons, to solutions of the finite period,  $2H$  and these are given exactly in terms of

cnoidal functions (Whitham [1974]). The structure functions,  $\phi(r)$ ,  $\gamma(r)$  and the wave speed  $d$  are determined (numerically) from a second order, ordinary differential eigenvalue problem.

For certain base columnar profiles  $d$  is negative and consequently the axisymmetric wave propagates upstream. Figure 2 is a plot of the streamlines (1.3) in a frame moving with the wave for such a case. The base columnar profile used here and throughout this paper is a purely swirling flow;  $W_0(r) \equiv 0$  and  $\alpha = 14$  in the notation of (1.1). The structure function  $\phi$  has been normalized so that  $\text{Max } \phi = 1$  and for this flow a recirculation zone (bubble) first appears in the streamline plot when the amplitude parameter,  $\epsilon$  reaches a value of 0.0155. For the value of  $\epsilon$  used here the plot is clearly reminiscent of the bubble type breakdown.

Our aim is to study the stability of the flows (1.3,4) to three dimensional disturbances viewing  $\epsilon$  as a bifurcation parameter. The analysis will be carried out in a frame moving with the wave, i.e. using the coordinates  $(r, x, \theta)$ . As the base flow is dependent on both the radial and axial variables,  $r$  and  $x$ , the stability equations separate only in the azimuthal variable,  $\theta$ . It will be in our interest also to study the stability of a related columnar flow that is constructed by taking the two dimensional flow (1.3,4) at a single axial station,  $x = 0$ , and extending it throughout the cylindrical domain without variation. For obvious reasons we will refer to this flow as the "mid-bubble" columnar flow and it is given explicitly as follows,

$$V_b(r) = V_0(r) + \frac{\epsilon}{r}\gamma(r), \quad (1.9)$$

$$W_b(r) = W_0(r) + \frac{\epsilon}{r}\phi'(r). \quad (1.10)$$

Plots of these profiles for various values of  $\epsilon$  are given in Figures 3 and 4. Provided the wavy flow (1.3,4) varies only slowly along the axis (as it will do if the period,  $2H$  of  $A(x)$  is very large), we can look on the midbubble profiles as models for the full two dimensional flow. We conjecture

that the stability of these columnar flows (the equations for which separate in both  $x$  and  $\theta$ ) should also be indicative of the stability properties of the full two dimensional flow.

In the rest of this paper we describe the numerical scheme used to solve the stability equations, we discuss its implementation and verification on the computer and finally we give results obtained for the stability of the midbubble columnar and the axisymmetric wavy flows presented above.

## 2. Numerical Methods.

For incompressible fluids the physical law of mass conservation reduces to the constraint that the velocity vector of the fluid be divergence free. The pressure is not then a thermodynamic variable determined by an equation of state but rather can be thought of as a Lagrange multiplier adjusting itself instantaneously to ensure that this kinematical constraint on the velocity vector is met. There is no evolution equation for the pressure nor does it satisfy any predetermined boundary or initial conditions.

Numericists, seeking to solve the governing equations approximately, have found that their greatest difficulty lies in the treatment of the pressure variable. While many ingenious methods have been devised to overcome the difficulties, the treatment advocated in this work is in a mathematical sense the most natural and offers many computational advantages. Here, the pressure term is eliminated from the equations entirely and the divergence free condition is satisfied exactly by the numerically obtained approximation to the velocity vector. Moreover, as the components of the velocity are expanded in terms of series of polynomials that arise as the solution to a singular Sturm Liouville problem, whose excellent approximation properties are well



documented (e.g. Gottleib & Orszag [1977], Quarteroni [1983]) convergence of our approximation will be bound only by the smoothness of the solution and by the number of terms used in the component expansions. For infinitely differentiable velocity fields we should expect to achieve "exponential convergence" (Canuto et al.)

The essence of the method (originally due to Leonard & Wray [1982]) involves expanding the velocity in a series of divergence free vector fields each of which satisfy the same boundary conditions as the velocity. The infinite sums are truncated and substituted into the governing equations, which are the Navier-Stokes or Euler equations linearized about the appropriate base flow. Inner products are taken with vectors fields which satisfy inviscid boundary conditions. This eliminates the pressure term from the equations and reduces the differential eigenvalue problem to a matrix eigenvalue problem. The eigenvalues determine the stability of the base flow and the eigenfunctions are the set of coefficients in the expansions of the corresponding perturbation velocity fields.

To examine how this method works we recall that it is well known (Ladyshenskaya [1966]) that  $L^2(D)$ , the space of square integrable vector functions defined on a bounded domain  $D$  ( $D \subset R^n$ ,  $n = 2,3$ ) can be decomposed into those that are divergence free and whose normal components vanish on the boundary and those that can be expressed as the gradient of a differentiable function defined on  $D$ . For this paper we will consider vector fields, defined on the section of a cylinder  $T$ , which are periodic in both the axial and azimuthal variables, having as their axial period the tube length,  $2H$ .

We will decompose  $L^2(T)$  as follows.

$$L^2(T) = J(T) + J^1(T), \quad (2.1)$$

where,

$$J(T) = \left\{ \begin{array}{ll} (a) \nabla \cdot \underline{u} = 0 \text{ in } T, \\ \underline{u} \in L^2(T) & (b) \underline{u} \cdot \underline{n} = 0 \text{ on } \partial T, \\ (c) \underline{u}|_{S_1} = \underline{u}|_{S_2} . \end{array} \right\} \quad (2.2)$$

$S_1$  and  $S_2$  represent the ends of the cylinder. Given in this form  $J(T)$  is clearly the space of

(a) incompressible, (b) inviscid, (c) periodic velocity fields.

The set of "viscous" velocity fields on  $T$  is a subset of  $J(T)$  denoted  $J^0(T)$ .

$$J^0(T) = \left\{ \underline{u} \in J(T) \mid \underline{u} = 0 \text{ on } \partial T \right\}. \quad (2.3)$$

An alternative representation of  $J(T)$  (Richtmyer [1978]) is given by,

$$J(T) = \left\{ \underline{u} \in L^2(T) \begin{array}{l} (a) \langle \underline{u}, \nabla p \rangle = 0 \text{ for all } p \in C^\infty(\overline{T}) \\ (b) \underline{u}|_{S_1} = \underline{u}|_{S_2} \end{array} \right\} \quad (2.4)$$

where  $\overline{T}$  is the closure of  $T$  and  $\langle \cdot, \cdot \rangle$  represents the usual inner product in  $L^2(T)$ ,

$$\langle \underline{u}, \underline{v} \rangle = \int_T \underline{u} \cdot \underline{v} \, r dr d\theta dx. \quad (2.5)$$

The space  $J(T)$  endowed with this inner product is a Hilbert space and a closed subspace of  $L^2(T)$ . The projection of  $L^2(T)$  onto  $J(T)$  will be denoted by  $\Pi$ . It is clear that vectors of the form  $\nabla p$  are perpendicular to all  $\underline{u}$  in  $J(T)$  and in fact (Ladyshenskaya [1966]),

$$J^\perp(T) = \left\{ \underline{u} \in L^2(T) \mid \underline{u} = \nabla p \text{ for some } p \text{ in } C^1(\overline{T}) \right\}. \quad (2.6)$$

$\Pi$  then has the following properties,

$$\Pi : L^2(T) \rightarrow J(T), \quad (2.7)$$

$$\Pi \underline{u} = \underline{u} \quad \text{for all } \underline{u} \in J(T), \quad (2.8)$$

$$\Pi \underline{\nabla} p = 0 \quad \text{for all } p \in C^1(\overline{T}). \quad (2.9)$$

To determine the linear stability of a flow  $\underline{U}$  to say, viscous disturbances which are periodic in  $x$  and  $\theta$  we consider whether infinitesimal perturbations to  $\underline{U}$  grow in time. Therefore we linearize the Navier-Stokes equations about  $\underline{U}$  and seek solutions in  $J^0(T)$  of the form,

$$\underline{u}(r, x, \theta) e^{-i\sigma t}. \quad (2.10)$$

The character of  $\sigma$  then determines the linear temporal stability of  $\underline{U}$ . If  $\sigma = \alpha + i\beta$  where  $\alpha, \beta$  are real then,

$$\begin{aligned} \beta > 0 & \Rightarrow \underline{U} \text{ is unstable,} \\ \beta = 0 & \Rightarrow \underline{U} \text{ is neutrally stable,.} \\ \beta < 0 & \Rightarrow \underline{U} \text{ is stable} \end{aligned} \quad (2.11)$$

The equations that must be solved have the form,

$$i\sigma \underline{u} = E\underline{u} + \text{Re}^{-1} S\underline{u}. \quad (2.12)$$

$E$  and  $S$  are operators defined on  $J(T)$  as follows,

$$S\underline{u} = -\Pi(\underline{\nabla} \times \underline{\omega}) \quad (2.13)$$

and

$$E\underline{u} = \Pi(\underline{\omega} \times \underline{U} + \underline{\Omega} \times \underline{u}), \quad (2.14)$$

where  $\underline{\Omega}$  and  $\underline{\omega}$  are respectively the base and perturbation vorticities, ( $\underline{\Omega} = \underline{\nabla} \times \underline{U}$ ,  $\underline{\omega} = \underline{\nabla} \times \underline{u}$ ).

Some suitable nondimensionalization has introduced the Reynolds number,

$$\text{Re} = \frac{U_0 R_0}{\nu}, \quad (2.15)$$

$R_0$  and  $U_0$  being characteristic length and velocity scales associated with the base flow and  $\nu$  is

the kinematic viscosity of the fluid. We can take  $R_0$  to be the radius of the tube and  $U_0$  to be the maximum value of the columnar base flow azimuthal velocity,  $V_0(r)$ .

The  $\nabla p$  term in the Navier-Stokes equation has been eliminated by projection onto  $J(T)$ . Projection of the stability equation onto a finite dimensional subspace,  $J_N(T)$  of  $J(T)$  is achieved in practice by taking the inner product of the equation with basis vectors for  $J_N(T)$ . This process eliminates the operator  $\Pi$  from the equation, for given any vector  $\underline{f}$  in  $L^2(T)$  and any vector  $\underline{\Delta}$  in  $J(T)$  we have that,

$$\langle \Pi \underline{f}, \underline{\Delta} \rangle = \langle \underline{f}, \underline{\Delta} \rangle, \quad (2.16)$$

as projections are self adjoint and as the projection of any vector in  $J(T)$  is itself.

It is worth emphasising that even when we are solving the viscous stability equations, we still project the governing equations onto the space of inviscid vector fields. The reason for this is that having found a velocity  $\underline{u}$  such that the vector  $\underline{f}$  defined as,

$$\underline{f} = i\sigma \underline{u} - \underline{\omega} \times \underline{U} - \underline{\Omega} \times \underline{u} + \nu \nabla \times \underline{\omega} \quad (2.17)$$

is orthogonal to all  $\underline{\Delta}$  in  $J(T)$  then  $\underline{f} \in J^\perp(T)$  and so there exists a scalar function  $p$  (a pressure) with  $\underline{f} = \nabla p$ . If, however,  $\underline{f}$  were in  $J^{0\perp}(T)$ , which contains  $J^\perp(T)$  then the existence of a pressure is not guaranteed and consequently  $\underline{u}$  may not correspond to a physical solution.

Leonard and Wray [1982] demonstrated a divergence free vector function expansion for viscous velocity fields, defined on a cylindrical domain that are Fourier decomposable in both the axial and azimuthal variables. We will construct a somewhat different set of basis vectors here.

The velocity field,  $\underline{u}$ , satisfies the continuity equation and is Fourier decomposable in  $x$  and  $\theta$ , which means in effect that only two of its three components,  $u, v, w$  are independent. This motivates the introduction of two vector families,  $\underline{\chi}_n^\pm$  in an expansion of the form,

$$\underline{u} = \sum_{nkm} \left\{ a_{2nkm} \underline{\chi}_n^+(r) + a_{2n-1km} \underline{\chi}_n^-(r) \right\} e^{i(kx + m\theta)} \quad (2.18)$$

The components of the vectors  $\underline{\chi}_n^\pm$  are found as follows. Expand two of the velocity components, say the first and the third, independently as,

$$u = \sum_{nkm} a_{2n-1km} f_n^-(r) e^{i(kx + m\theta)}, \quad (2.19)$$

$$w = \sum_{nkm} a_{2nkm} f_n^+(r) e^{i(kx + m\theta)}, \quad (2.20)$$

where  $f_n^\pm(r)$  are complete sets of polynomials chosen to satisfy the boundary conditions that are imposed on  $u, w$ . The  $r$  and  $x$  components of  $\underline{\chi}_n^\pm$  have now been picked and it remains for us to choose the  $\theta$  components in a manner that ensures the vectors  $\underline{\chi}_n^\pm e^{i(kx + m\theta)}$  are divergence free. Consider for example,  $\underline{\chi}_n^-(r)$ .

$$\nabla \cdot \left( \underline{\chi}_n^-(r) e^{i(kx + m\theta)} \right) = 0, \quad (2.21)$$

$\Rightarrow$

$$(rf_n^-(r))' + im\chi_{n,\theta}^- = 0, \quad (2.22)$$

where the prime denotes a derivative with respect to  $r$ . This equation gives us the  $\theta$  component of  $\underline{\chi}_n^-(r)$ . Rescaling, it is found that an expansion of the form (2.14) is possible for non-zero azimuthal wavenumbers where,

$$\underline{\chi}_n^-(r) = \left( imf_n^-(r), -(rf_n^-(r))', 0 \right), \quad (2.23)$$

$$\underline{\chi}_n^+(r) = \left( 0, -rkf_n^+(r), mf_n^+(r) \right) \quad (2.24)$$

and such an expansion will guarantee that  $\underline{u}$  is divergence free. This expansion is clearly incomplete for azimuthal wave number zero, ( $m = 0$ ), i.e. for axisymmetric flows. For that case the following expansion vectors can be used.

$$\underline{\chi}_n^-(r) = \left( ikf_n^-(r), 0, -\frac{1}{r} (rf_n^-(r))' \right), \quad (2.25)$$

$$\underline{\chi}_n^+(r) = \left( 0, f_n^+(r), 0 \right) \quad (2.26)$$

The polynomials  $f_n^\pm(r)$  must be chosen so that the vector  $\underline{u}$  given by (2.14) satisfies appropriate (viscous or inviscid) boundary conditions on the walls of the domain,  $T$  and is single valued at the origin,  $r = 0$ . We will denote the polynomials used in the inviscid case by  $a_n^\pm(r)$  reserving  $f_n^\pm(r)$  for viscous expansions. We have then upon truncating (2.14) an approximation to  $\underline{u}$  of the form,

$$\underline{u}_{NKM} = \sum_{n=1}^N \sum_{k=-K}^K \sum_{m=-M}^M a_{nkm} \underline{D}_{nkm}(r, x, \theta) \quad (2.27)$$

where,

$$\underline{D}_{nkm}(r, x, \theta) = \underline{\chi}_n^\pm(r; k, m) e^{i(kx + m\theta)}. \quad (2.28)$$

The projection vectors will have the same form as the expansion vectors, i.e. we will project with vectors,  $\underline{\Delta}_{lpq}(r, x, \theta)$ , where

$$\underline{\Delta}_{lpq}(r, x, \theta) = \underline{\xi}_l^\pm(r; k, m) e^{i(pz + q\theta)} \quad (2.29)$$

for

$$l = 1, \dots, N; \quad p = -K, \dots, K; \quad q = -M, \dots, M$$

with the vectors  $\underline{\xi}_l^\pm$  being given by equations (2.23 - 26) using the inviscid polynomials,  $a_l^\pm(r)$  for

the components.

It is possible to choose the polynomials  $a_l^\pm(r)$  and  $f_n^\pm(r)$  in many different ways. Leonard & Wray [1982] in their consideration of certain turbulence simulations employed an unusual set of Jacobi polynomials to reduce the bandwidth of the final matrix system. These polynomials were also used by Spalart [1983] in his simulation of boundary-layer transition. Moser & Moin [1984] in their work on the infinite Taylor Couette system, used Tchebychev polynomials and incorporated the weight function, against which these polynomials are orthogonal, into the projection vectors. Here, we will construct the basis vectors from Legendre polynomials. All of the above sets are solutions to singular Sturm Liouville problems and consequently we can expect expansions in terms of any of these polynomials to exhibit excellent convergence properties.

The single valuedness criterion, which must be applied along the centre line of the tube for the vector  $\underline{u}$ , causes the polynomials  $f_n^\pm(r)$  and  $a_l^\pm(r)$  to depend on  $m$ , the azimuthal wavenumber (Joseph [1970]). One appropriate choice for  $a_l^\pm(r)$  is,

$$\begin{aligned} a_l^+(r) &= rP_l(2r-1) && \text{for all } m, \\ a_l^-(r) &= (1-r)P_l(2r-1) && \text{if } |m| = 1, \\ a_l^-(r) &= r(1-r)P_l(2r-1) && \text{if } |m| \neq 1, \end{aligned} \tag{2.30}$$

where the radial variable has been scaled by the tube radius and  $P_l(r)$  is the Legendre polynomial of order  $l$  (Abramowitz & Stegun [1970]). The corresponding choice for  $f_n^\pm(r)$  is,

$$\begin{aligned} f_n^+(r) &= r(1-r)P_n(2r-1) && \text{for all } m, \\ f_n^-(r) &= (1-r)^2P_n(2r-1) && \text{if } |m| = 1, \\ f_n^-(r) &= r(1-r)^2P_n(2r-1) && \text{if } |m| \neq 1. \end{aligned} \tag{2.31}$$

As all of our stability problems separate in the azimuthal direction, this dependence on  $m$  presents no difficulty. We solve separate problems for each azimuthal wavenumber; so having chosen an  $m$  the expansion and projection sets are fixed throughout the calculation. Indeed, in theory there is no difficulty even if the problem at hand is truly three dimensional; however some care is required in implementing the method to ensure that the correct radial polynomial set is being used for each azimuthal component of the velocity.

### 3. Implementation and Verification of the Method.

Equation (2.12) is solved approximately by using the expansion  $\underline{u}_{NKM}$  for  $\underline{u}$  and taking inner products of the equation with the projection vectors,  $\underline{\Delta}_{pq}$  to get a generalized matrix eigen problem for the eigenvalues  $\sigma$  and the eigenvectors  $a$  (the coefficients in the expansion  $\underline{u}_{NKM}$ ). This matrix problem can be written as,

$$\sigma \bar{A} a = \left( \bar{B} + \frac{i}{\text{Re}} \bar{C} \right) a. \quad (3.1)$$

The matrix  $\bar{A}$  is purely real and arises from the fact that the expansion and projection vectors are not orthonormal.

$$\bar{A}_{lnpqkm} = \langle \underline{\Delta}_{lpq}, \underline{D}_{nkm} \rangle, = \langle \underline{\xi}_l, \underline{\chi}_n \rangle \delta_{pk} \delta_{qm}, \equiv A_{ln} \delta_{pk} \delta_{qm}, \quad (3.2)$$

The Kronecker delta symbol,  $\delta_{ij}$ , arises because the Fourier bases employed in the axial and azimuthal directions are orthogonal. The matrix  $\bar{B}$  arising from the convection terms is also purely real.

$$\bar{B}_{lnpqkm} = \langle \underline{\Delta}_{lpq}, (\underline{\nabla} \times \underline{D}_{nkm}) \times \underline{U} + \underline{\Omega} \times \underline{D}_{nkm} \rangle. \quad (3.3)$$

Finally, the matrix  $\bar{C}$  arising from the viscous terms is purely imaginary.



$$\overline{C}_{lnpkqm} = \langle \underline{\Delta}_{lpq}, \underline{\nabla} \times \underline{\nabla} \times \underline{D}_{nkm} \rangle. \quad (3.4)$$

Using the orthogonality of the Fourier bases it can be written as,

$$\overline{C}_{lnpkqm} = C_{ln} \delta_{pk} \delta_{qm}. \quad (3.5)$$

The form of the matrix  $\overline{B}$  depends on the base flow  $\underline{U}$ . For columnar flows which are independent of  $x$  and  $\theta$  it is possible to find a matrix  $B$  such that,

$$\overline{B}_{lnpkqm} = B_{ln} \delta_{pk} \delta_{qm}. \quad (3.6)$$

The stability of these flows can be determined by solving the  $O(N)$  generalized matrix eigen problem,

$$\sigma Aa = \left( B + \frac{i}{\text{Re}} C \right) a. \quad (3.7)$$

On the other hand, for the axisymmetric wavy base flow (1.3,4) we have

$$\overline{B}_{lnpkqm} = \hat{B}_{lnpk} \delta_{qm} \quad (3.8)$$

and the stability equation is the  $O((2K+1) \times N)$  matrix equation,

$$\sigma \hat{A}a = \left( \hat{B} + \frac{i}{\text{Re}} \hat{C} \right) a, \quad (3.9)$$

where  $\hat{A}$  and  $\hat{C}$  are the  $O((2K+1) \times N)$  block diagonal matrices  $A_{ln} \delta_{pk}$  and  $C_{ln} \delta_{pk}$  respectively. The matrices depend parameterically on the wavenumbers of the projection and expansion vectors so we separate them into submatrices that can be evaluated independently of these and the other parameters (in particular  $\epsilon$ ) occurring in the base flow. The submatrices are evaluated once and then stored in the computer. The required integrations can be done at very little cost by utilizing the orthogonality properties of the expansion and projection polynomials. The full matrices are then be reassembled without the need for doing any further integrations.

One can always band the  $A$  and  $C$  matrices by appropriate choice of the polynomials  $f_n^\pm(r)$  and  $a_i^\pm(r)$ . However the matrix  $B$  will generally be full, though for certain rather simple base flows such as the Hagen Poiseuille flow considered by Leonard & Wray [1982] it is also possible to band  $B$ . The matrix  $A$  was inverted to produce a regular eigenvalue problem in place of (3.1) and the  $QR$  algorithm was used to extract the eigenvalues. We also note that the matrix problem we get when considering the inviscid stability of base flows is a purely real one and consequently the eigenvalues occur, as they should do, in conjugate pairs.

A computer code has been written which implements the method we have been describing to solve the stability problems, both viscous and inviscid, for all columnar flows and for axisymmetric flows of the form (1.3,4). Both the direct and adjoint versions of the stability problems can be solved. The adjoint viscous stability problem is to find a  $\underline{u}$  in  $J^0(T)$  such that

$$i\lambda \underline{u} = E^* \underline{u} + \frac{1}{\text{Re}} S \underline{u}. \quad (3.10)$$

The operator  $E^*$  is the adjoint operator to  $E$  and is given by,

$$E^* \underline{u} = -\Pi \left( \underline{\Omega} \times \underline{u} + \underline{\nabla} \times (\underline{u} \times \underline{U}) \right) \quad (3.11)$$

The direct and adjoint spectra obtained by solving (2.12) and (3.10) should, of course, be conjugate to each other and how well a numerical scheme reproduces this theoretical result is a test of its accuracy.

We verified the code by calculating the stability of rotating Poiseuille flow,

$$\underline{U} = \left( 0, \quad V_1 r, \quad W_1 (1 - r^2) \right) \quad (3.12)$$

Cotton et al. [1980] found that this flow with  $V_1 = 0.2147$  and  $W_1 = 1.0$  was neutrally stable to disturbances having azimuthal wavenumber,  $m = 1$  and axial wavenumber,  $k = -1$  for a

Reynolds number of 156. The following table lists the most unstable eigenvalue we found for this flow with the same wavenumber pair for the disturbance. The first column of the table gives  $N$ , the number of radial basis vectors that were used to obtain the eigenvalue given in the next two columns,  $N$  is also the order of the matrix problem that needs to be solved at each step.

Most unstable eigenvalue found for the rotating Poiseuille flow (3.12) with $m = 1$ , $k = -1$ , $V_1 = 0.2147$ , $Re = 156.0$ .		
$N$	frequency	growth rate
4	-0.00029	.00334
6	-0.00279	.00101
10	-0.00284	.000001
14	-0.002847	.0000001
18	-0.002847898	.0000001379
22	-0.002847898	.0000001378

The convergence is exponential in  $N$  or some power of  $N$  and there is no evidence of significant roundoff error. The following table lists the corresponding eigenvalue found by solving the adjoint viscous stability problem for the same wavenumber pair and baseflow.

Eigenvalue found by doing the adjoint viscous stability problem for the flow (3.12), with $m = 1$ , $k = -1$ , $V_1 = 0.2147$ , $Re = 156.0$ .		
$N$	frequency	growth rate
4	-0.00270	-.000094
6	-0.00280	-.000013
10	-0.00284	-.000004
14	-0.002847	-.0000002
18	-0.002847898	-.0000001379
22	-0.002847898	-.0000001379

Clearly the agreement between the adjoint and direct results is excellent. Inviscid stability results for flows of the form (3.12), obtained using our code also compare well with results in the

literature. These results instill confidence in the accuracy of the numerical method and in the code that implements it, at least for the case of columnar flows.

#### 4. Stability Results for the Vortex Breakdown Model Flows.

In this section we will present the results obtained to date for the stability of the midbubble columnar, (1.9,10) and the wavy vortex (1.3,4) flows. Although these flows are inviscid we will consider their stability to both viscous and inviscid disturbances (i.e. we will solve the linearized Euler and the linearized Navier-Stokes equations for these flows). The justification for doing a viscous analysis is that the "real" flow is of course, viscous. Moreover, the inclusion of the higher order dissipative terms eliminates certain technical difficulties that arise due to the existence of critical layers in the neutrally stable eigenfunctions for columnar flows (Drazin & Reid [1981]).

We will begin by presenting the viscous results for the midbubble columnar flows. We found that thirty radial vector modes ( $N = 30$ ) were adequate to resolve the most unstable eigenmode (i.e. the mode whose eigenvalue had the largest imaginary part) to three decimal places for these flows at low Reynolds numbers and that this number increased as the Reynolds number grew. Frequent checks were carried out on the accuracy of the computed eigenvalues both by increasing the order of the expansion and also by computing the adjoint spectrum for the same set of flow parameters. The difference between the most unstable eigenvalue as computed by the direct and adjoint versions of the code was always less than 1%.

Having fixed the number of radial expansion vectors in our system the viscous eigenvalues for the midbubble flows depend on four parameters,

$$\sigma = \sigma(m, k, \epsilon, Re). \quad (4.1)$$

The base columnar flow ( $\epsilon = 0$ ) was found to be stable to all disturbances. It seems that even for very small values of  $\epsilon$  (values for which there is no recirculation zone in the full two dimensional flow) the midbubble flows are unstable. This is documented in the following table which gives bracketing values for the critical Reynolds number for various values of  $\epsilon$ .

Bracketing values for the critical Reynolds number. Various values of $\epsilon$ and $m = -1$ .		
$\epsilon$	Stable for Re	Unstable for Re
0.000	Stable for all Re	
0.005	550	600
0.010	180	200
0.015	160	180
0.020	110	120
0.025	60	80
0.030	42	45

For large enough values of  $\epsilon$  disturbances having both positive and negative azimuthal wavenumbers can destabilize the midbubble flows with the negative azimuthal modes giving rise in general to the largest values for the growth rates. In particular disturbances with azimuthal wavenumber,  $m = -1$  were found to be the most dangerous. This is shown in the following table.

Bracketing values for the critical Reynolds number. Various values of $m$ with $\epsilon = .03$ .		
$m$	Stable for Re	Unstable for Re
-1	42	45
-2	90	100
-3	700	800
-4	1200	1400
-5	1400	1600

For fixed values of  $m$  and  $\epsilon$ , a two parameter ( $k$ , Re) study was carried out. With  $\epsilon = .03$  and  $m = -1$  we obtain the stability diagram shown in Figure 5. The stability boundary appears to be a parabolic curve which is markedly asymmetric with respect to the  $k = 0$  line. Within this curve the base flow is unstable to a range of axial wavenumbers; however, there is a "tongue" of stable wavenumbers that gradually thins out as the Reynolds number is increased. The  $k = 1$  mode is the final one to be excited, this does not happen until  $Re = 4500$  (approximately).

We now consider the stability of the midbubble flows to inviscid disturbances. The inviscid stability of columnar flows is governed by an ordinary differential equation, the Howard-Gupta [1962] equation. A number of analytic results obtained from this equation exist in the literature. Leibovich & Stewartson [1982] showed that a sufficient condition for the instability of a columnar flow is that the function,

$$F(r) \equiv V(r)\Lambda'(r)\left(\Lambda'(r)\Gamma(r) + W'(r)^2\right) \quad (4.2)$$

be negative somewhere in the domain of interest. ( $\Lambda$  is the angular velocity,  $\frac{1}{r}V$  and  $\Gamma$  is the circulation  $rV$ .)

The function  $F(r)$  is easily evaluated for the midbubble profiles and this has been done for a range of values of  $\epsilon$ . It was found that  $F$  first becomes negative only when a critical value of  $\epsilon$  is reached,  $\epsilon = 0.132$ . Consequently the midbubble flows are guaranteed to be unstable by the Leibovich-Stewartson criterion for values of the parameter  $\epsilon$  slightly below those needed to produce a recirculation region in the full two dimensional flow (recall this happens for  $\epsilon = 0.155$ ).

A normal mode stability analysis was carried using the divergence free expansion method and the numerically obtained results confirm and somewhat extend the predictions of the Leibovich-Stewartson criterion. Once again we found that the disturbances giving rise to the largest growth rates had azimuthal wavenumbers,  $|m| = 1$ . Figure 6 shows how the maximum growth rate varies (almost linearly) with the amplitude parameter,  $\epsilon$ . In the figure we can see that the normal mode analysis extends the previously obtained results in that midbubble flows with  $\epsilon < 0.0132$  are also found to be unstable, even though  $F(r) > 0$  for these flows. The maximum growth rates also increase with the size of the axial wavenumber,  $|k|$ , as shown in Figure 7.

We conclude that the midbubble flows are definitely unstable on both viscous and inviscid grounds for values of the parameter,  $\epsilon$  below those needed to produce a reversed flow region in the full two dimensional wavy flow. Moreover the most destabilizing disturbances have azimuthal wavenumbers,  $|m| = 1$  and short axial period, ( $|k| > 1$ ). The unstable inviscid eigenfunctions tended to have regions of steep gradient near the origin and this made their resolution difficult.

While these midbubble stability results we have just reported tend to support the conjectures made about vortex breakdown, they are not encouraging for the numericist seeking to investigate the stability of the cnoidal wave flows (1.3,4). As we noted earlier, the stability equation

for these flows do not separate in  $x$  and consequently we have to solve  $O(N(2K+1))$  matrix eigenvalue problems for each flow. It would seem to be necessary to include modes having a short axial period in our expansion,  $u_{NKM}$  which means that  $K$  will be large. The inclusion of these modes is also dictated by physical considerations. We should like the disturbance to include modes that scale with the dimensions of the bubble and in fact visualization studies indicate that the asymmetric unstable modes do have short axial periods. However, our experience with the midbubble flows show that these unstable modes are difficult to resolve radially, consequently the number of radial vector functions,  $N$  in our expansion must also be large.

With these constraints the normal mode analysis of the cnoidal wave flows becomes prohibitively expensive, (recall that the number of operations needed to extract all the eigenvalues of a matrix is proportional to the cube of its order). To alleviate the cost problems most of the runs for these flows were done with the axial period of the cnoidal wave fixed at 1 ( $H = 0.5$ , units are tube radii). While such short period flows violate the assumptions needed to produce the solutions (1.3,4) it was hoped that these flows would be unstable to disturbances with smaller values for  $K$ . Indeed convergence studies indicate that adequate resolution in the axial direction was obtained with  $K \approx 10$ . Up to 40 radial modes were used in the study, leading to an  $O(840)$  matrix eigenvalue problem when  $K = 10$ . (The calculations were performed on an Floating Point Systems 164 series vector processor at Cornell University, using a vectorized version of the  $QR$  algorithm, optimized for this machine and using some 16 megabytes of memory.) The adjoint problem was also solved on each run and only those eigenvalues which agreed well between the adjoint and direct calculations were considered.

Unfortunately the short period flows behave less like the midbubble flows and more like the underlying, stable columnar flows. This is indicated in Figure 8 which plots the least stable



growth rates found for the various values of  $\epsilon$  as the Reynolds number is increased ( $m = -1$  in this plot). All these short period cnoidal wave flows are stable, though marginally so. The disturbances with  $|m| = 1$  are again the most unstable. Figure 9 shows the least stable growth rates found for some other azimuthal wavenumbers. The inviscid stability runs that were performed also failed to turn up any definite evidence of instability for these flows.

We have considered the viscous and inviscid stability of some axisymmetric flows which are said to model the bubble type of vortex breakdown. The investigation was carried out by expanding the perturbation velocity in terms of the new set of divergence free vectors presented in section 2. The stability results for the midbubble flows support the conjectured mechanism for breakdown and it was found that the most dangerous disturbances have a short axial period and azimuthal wavenumbers,  $|m| = 1$ . The two dimensional flows we considered all had rather short axial periods and these do not model the physical phenomena well. No conclusions as to the stability of these flows can be drawn because the normal mode analysis used here can only prove instability (by finding a growing disturbance). No evidence of instability was found but this may well be because we failed to include enough axial modes in our expansion for the disturbance. The study clearly points out that linear stability investigations for complex base flows are far from trivial from a computational point of view.

## 5. Acknowledgements

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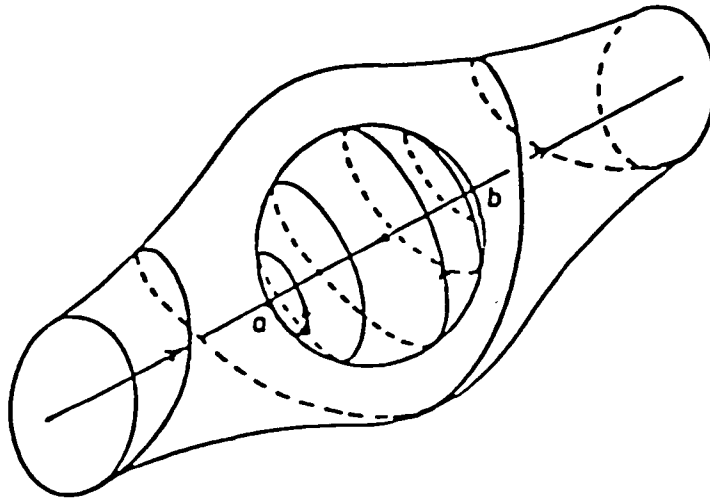


Figure 1. Axisymmetric bubble type vortex breakdown (after Leibovich [1978]).

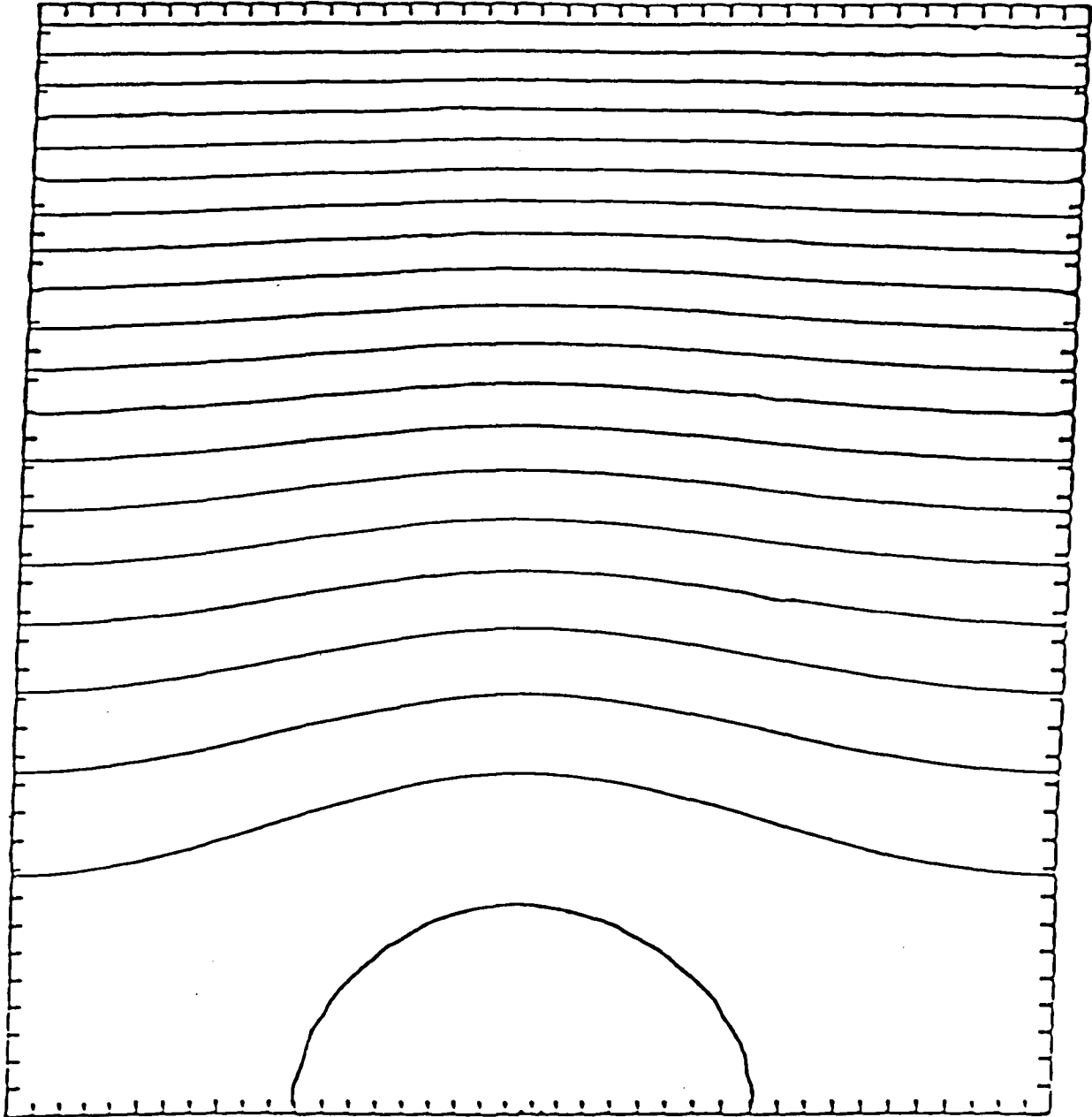


Figure 2. Streamlines (1.3),  $\epsilon = .02$ ,  $H = \pi$ .

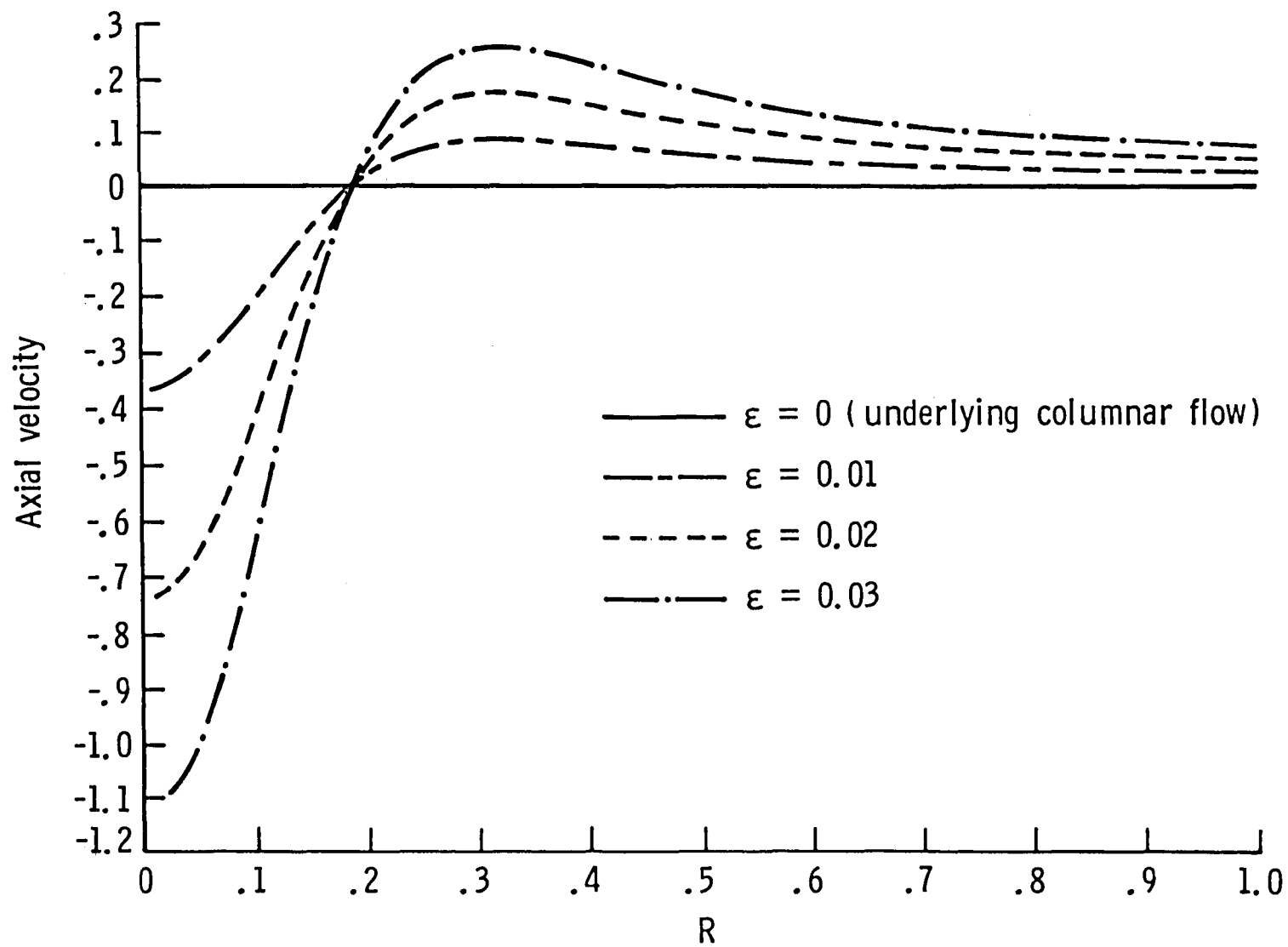


Figure 3. Axial velocity profiles for the midbubble flow.

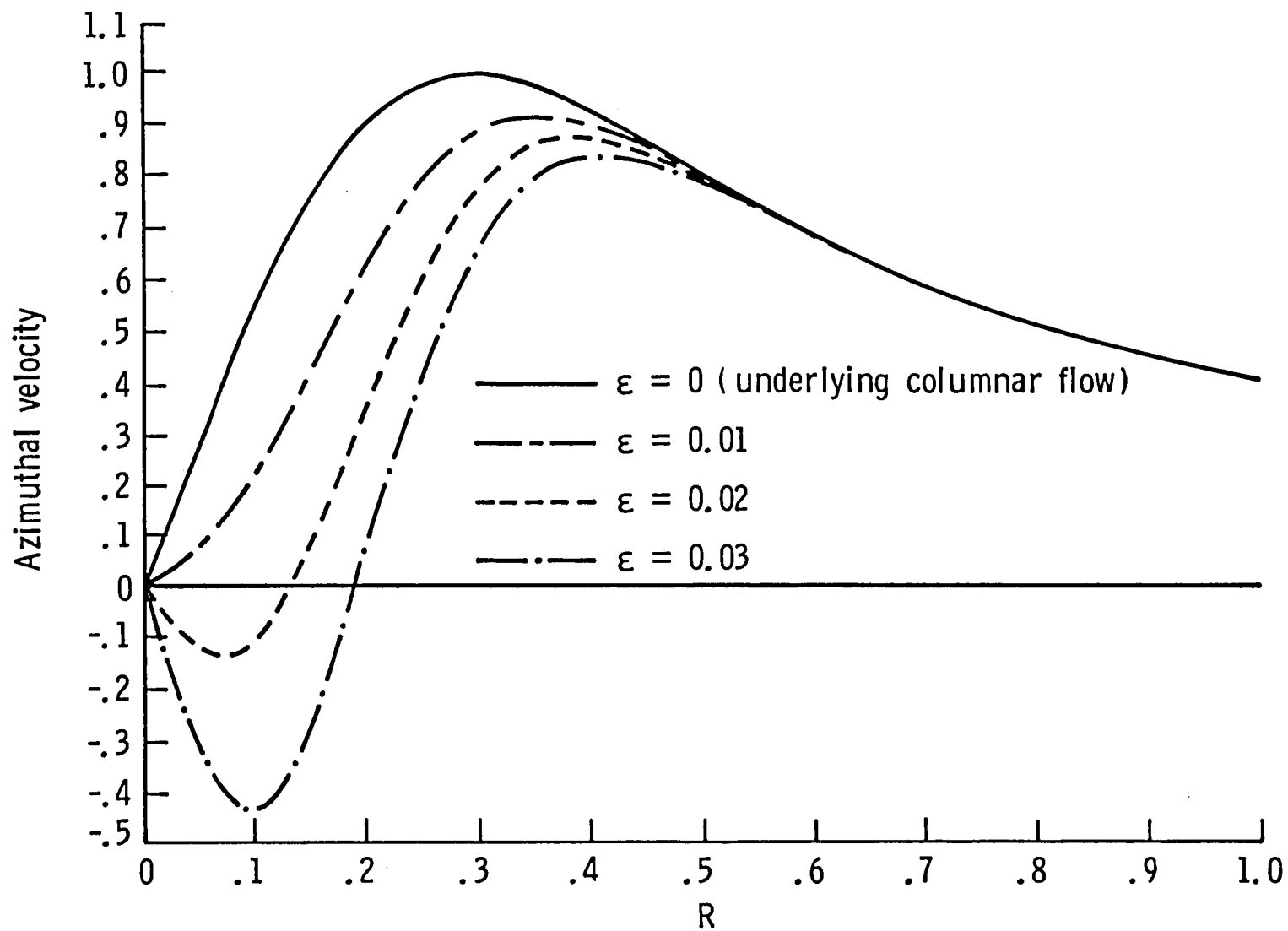


Figure 4. Azimuthal velocities for the midbubble flows.

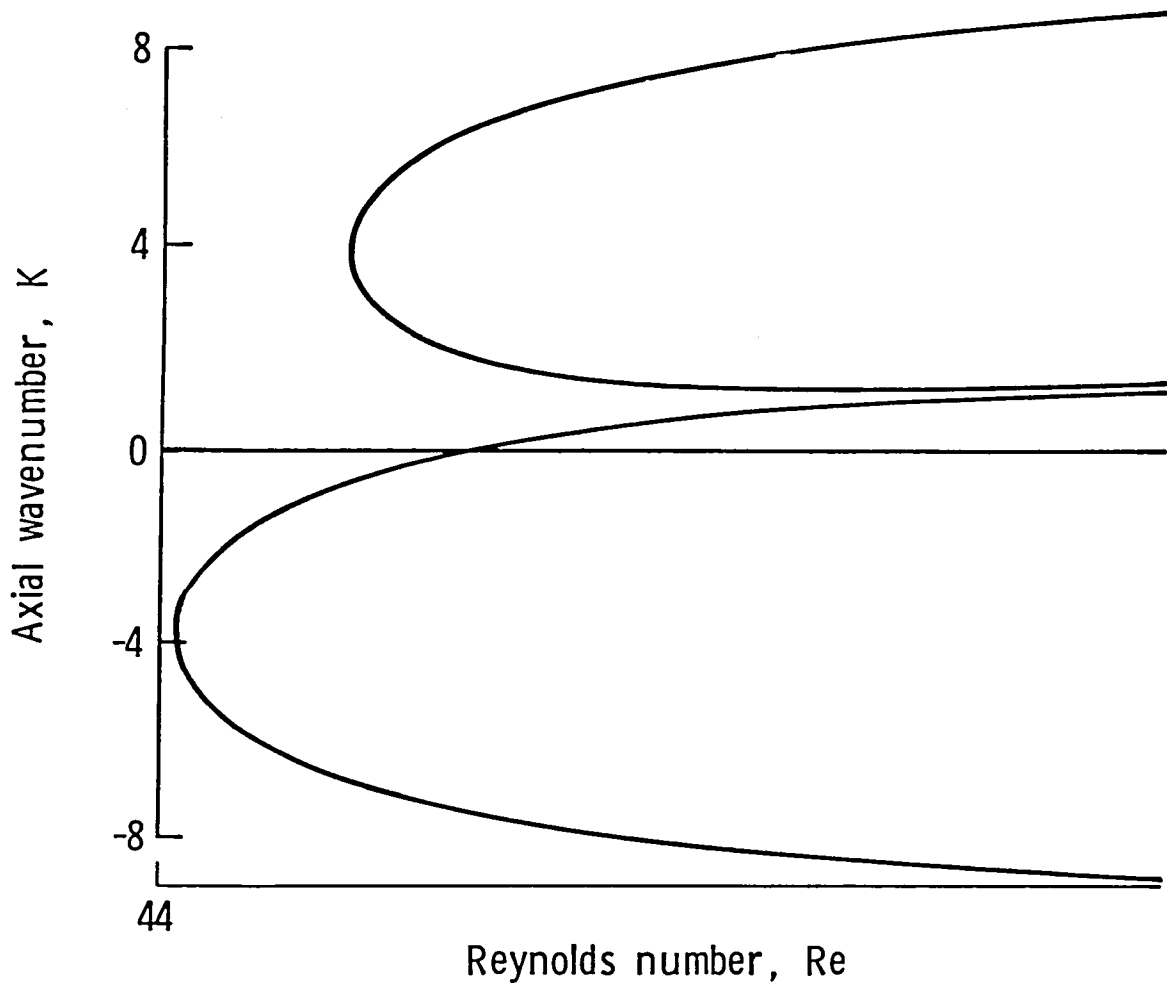


Figure 5. Stability diagram for the midbubble flow with  $\epsilon = .03$  and  $m = -1$ .



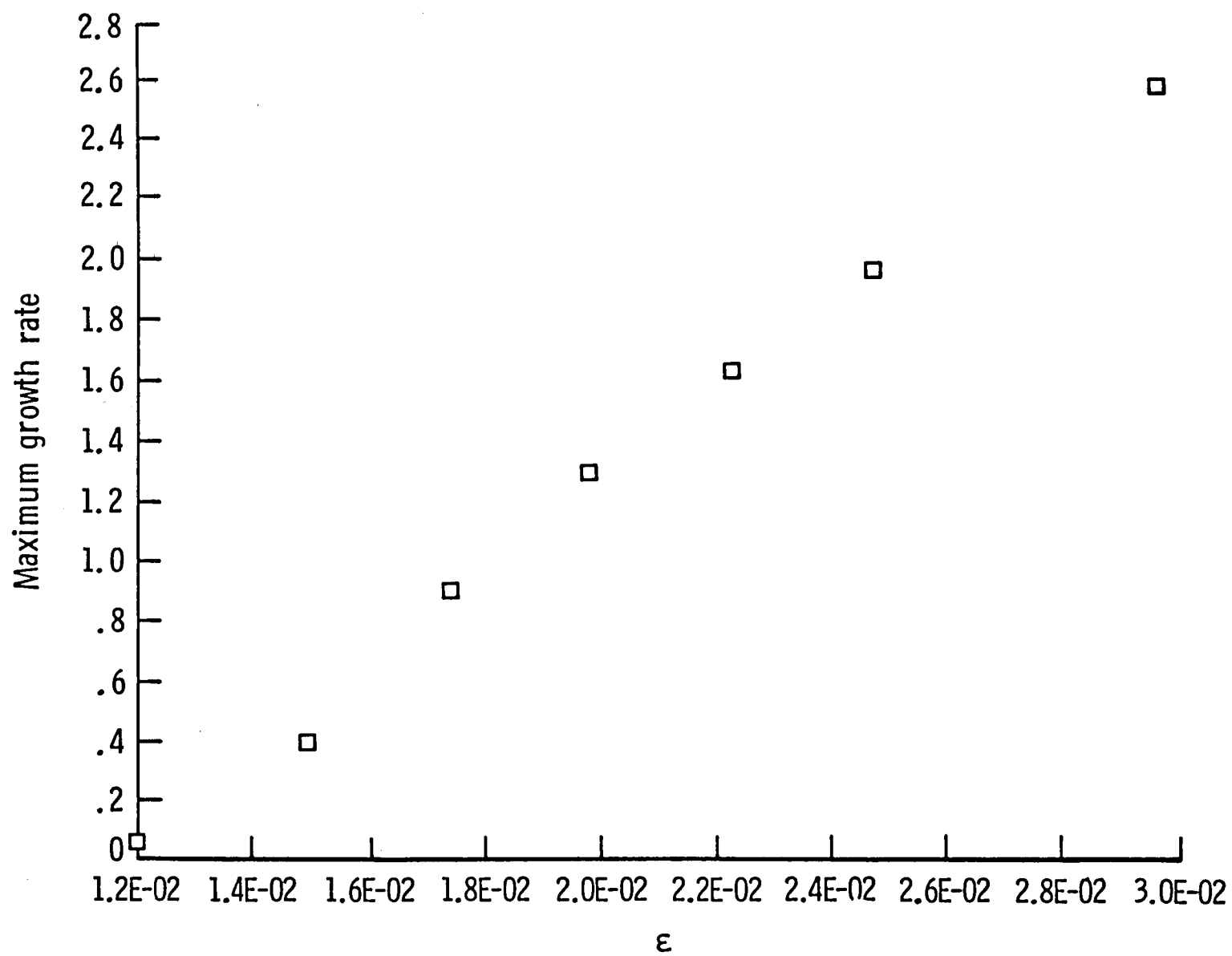


Figure 6. Maximum growth rates plotted against  $\epsilon$ ,  $m = -1$  and  $k = 6$ .

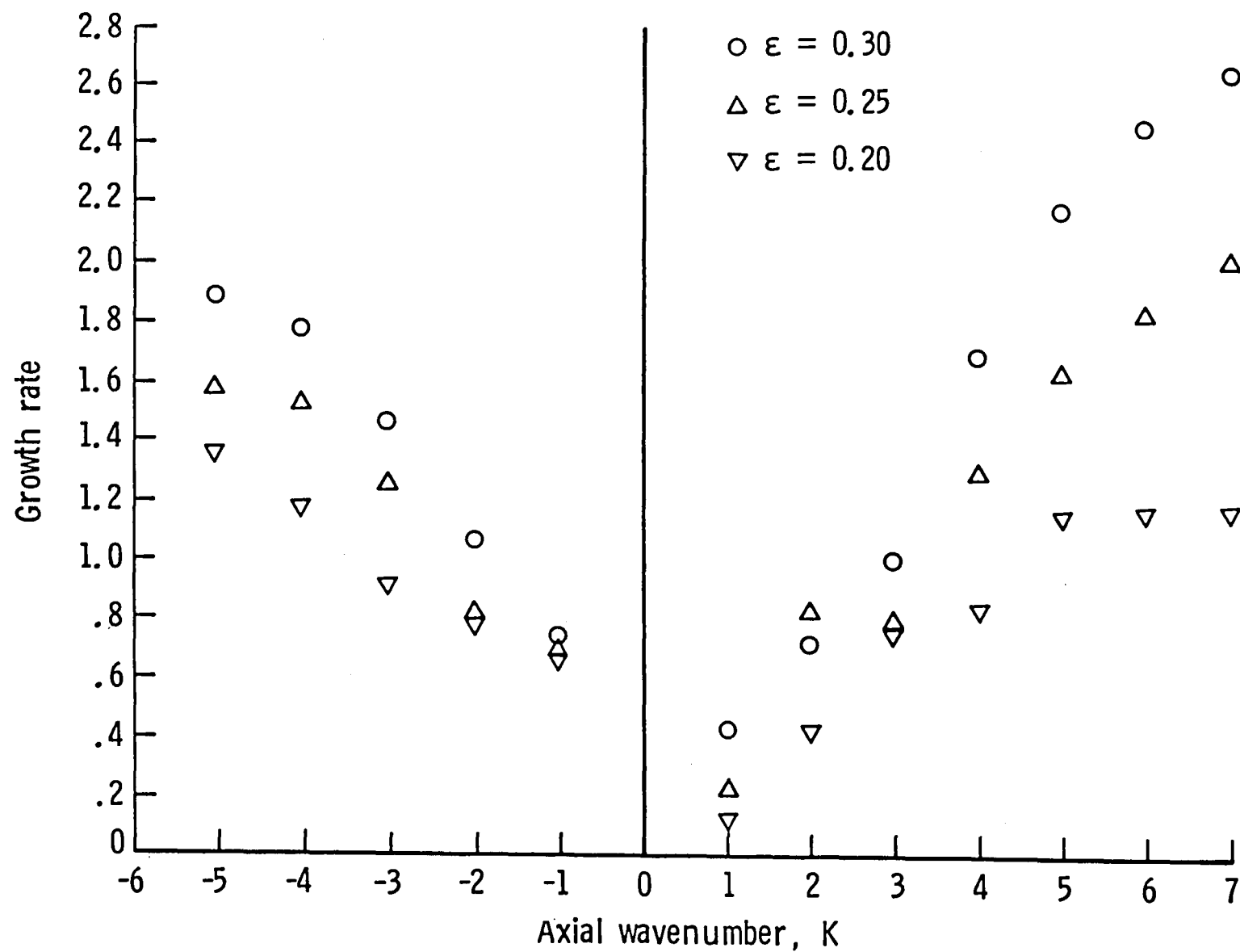


Figure 7. Maximum growth rates plotted against axial wavenumber, (inviscid analysis for the midbubble flow),  $m = -1$ .

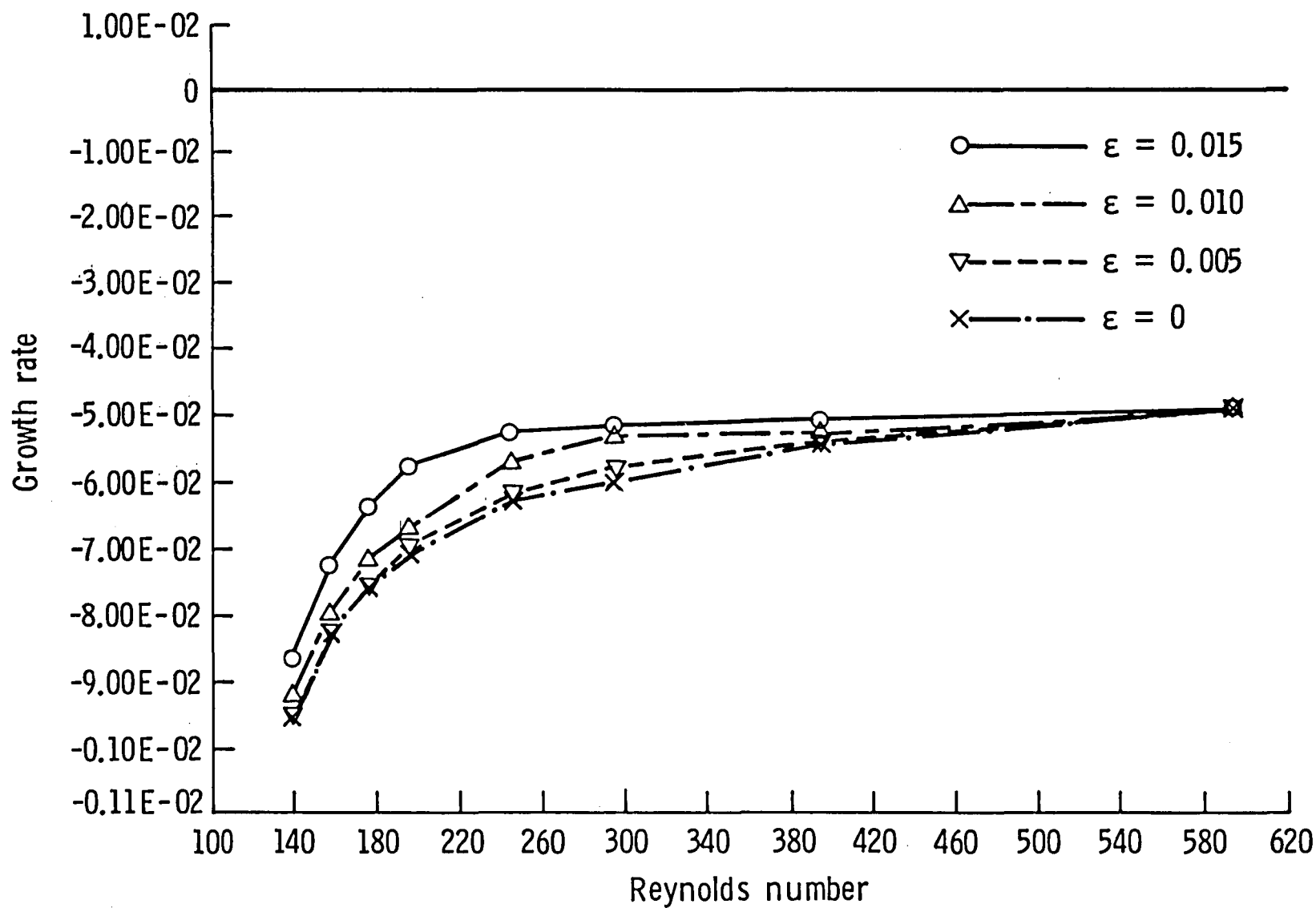


Figure 8. Least stable growth rates found for the full two dimensional flow (1.3,4),  
 $H = .5$ ,  $m = -1$ .

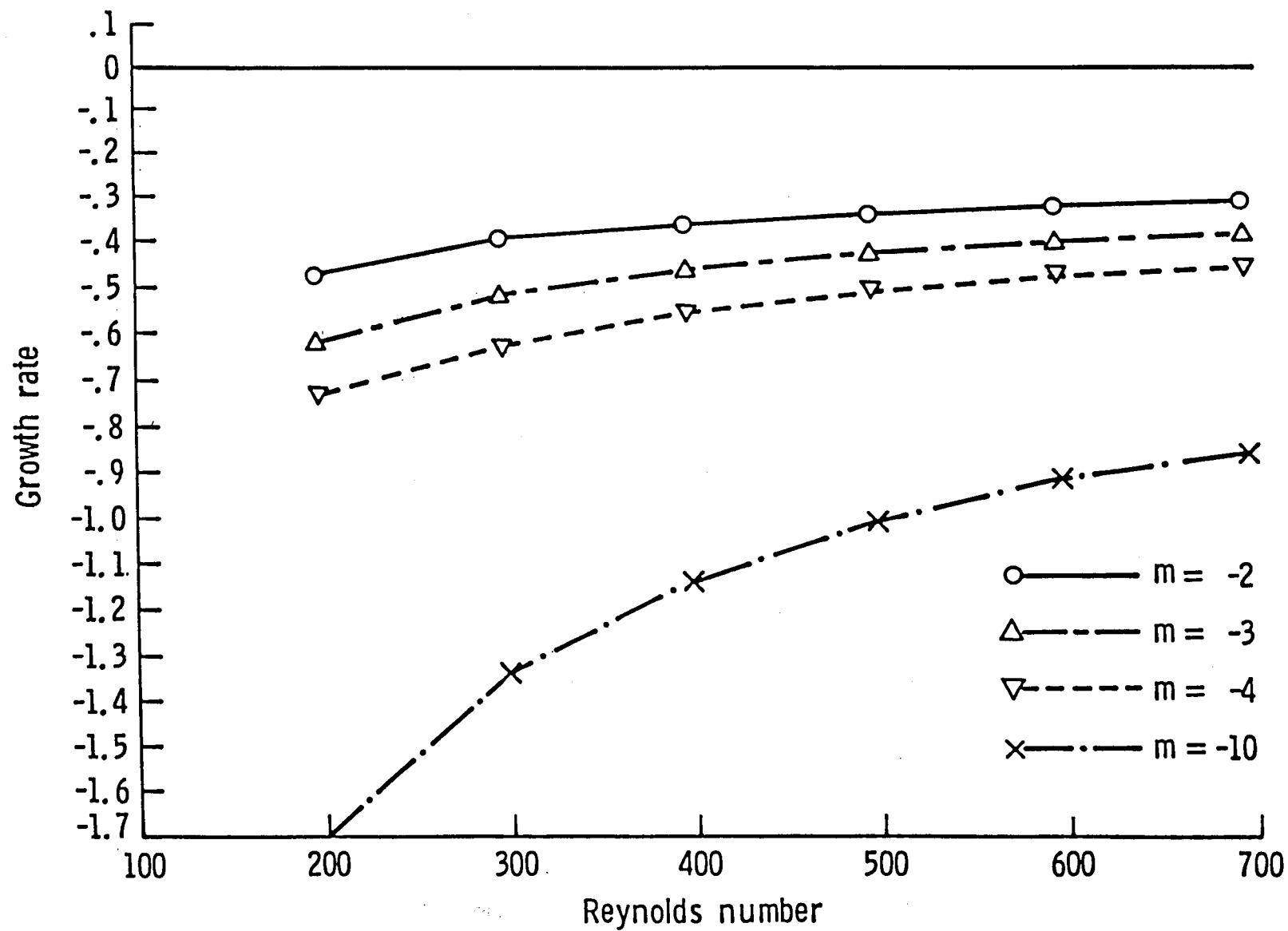


Figure 9. Least stable growth rates found for various values of  $m$ ,  $H = 0.5$ ,  $m = -1$ .

# Standard Bibliographic Page

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16. Abstract  The term "vortex breakdown" refers to the abrupt and drastic changes of structure that can sometimes occur in swirling flows. It has been conjectured that the "bubble" type of breakdown can be viewed as an axisymmetric wave travelling upstream in a primarily columnar vortex flow. In this scenario the wave's upstream progress is impeded only when it reaches a critical amplitude and it loses stability to some axisymmetric wavy flows, which model vortex breakdown, to three dimensional disturbances viewing the amplitude of the wave as a bifurcation parameter. We will also look at the stability of a set of related, columnar vortex flows which are constructed by taking the two dimensional flow at a single axial location and extending it throughout the domain without variation. The method of our investigation will be to expand the perturbation velocity in a series of divergence free vectors which ensures that the continuity equation for the incompressible fluid is satisfied exactly by the computed velocity field. Projections of the stability equation onto the space of inviscid vector fields eliminates the pressure term from the equation and reduces the differential eigen problem to a generalized matrix eigen problem. Results are presented both for the one dimensional, columnar vortex flows and also for the wavy "bubble" flows.					
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